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ON THE MOTION OF CONTROLLABLE MECHANICAL SYSTEMS

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The motions of mechanical systems depend upon the forces action and the constraints imposed, thanks to which it is possible to control the system's motion both with the aid of forces (dynamic control) as well as with the aid of constraints (kinematic control). Holonomic and linear nonholonomic constraints, depending upon certain variable parameters, apparently were first examined in a series of papers by Grdina on the dynamics of living organisms (for brevity we specify only the papers [1, 2] from this series, in which the remaining papers are mentioned). Having taken conditions on the virtual displacements as the parametric constraints and having adopted an axiom for the determination of the ideal constraints analogous to those for the usual constraints, Grdina constructed, on the basis of the d'Alembert-Lagrange principle, the foundations of the analytical dynamics of systems with parametric constraints. In particular, for systems with parametric constraints he showed the validity of all the fundamental types of equations of motion of systems with ordinary constraints and showed that these equations can be derived from the Gauss principle. Many years later a number of these results, with certain extensions, were published by Kirgetov [3, 4] who was apparently unaware of Grdina's papers. Together with this a modified Gauss principle was given in [3] for systems with holonomic parametric constraints, equivalent to the latter under the usual constraints, and the statement that "the Gauss principle is lacking in the case of systems with parametric constraints" was retained, contradicting the results in [1].

In the present paper we carry on the investigation of the motion of controlled systems. For parametric holonomic and nonlinear nonholonomic constraints we extend the definition, proposed in [5] for ordinary constraints, of virtual displacements, leading in a natural way to Chetaev's conditions [6]. From the d'Alembert-Lagrange principle we derive other fundamental variational differential principles of the dynamics of the controlled systems, namely, the Jourdin principle, the Mach inequalities and the generalized Gauss principle, the Chetaev principle. We have shown that for the controlled system all these differential principles are applicable and equivalent, as in the case of uncontrolled systems [5]. From the equations of motion it follows that in the final analysis a kinematic control reduces to a dynamic control. In the conclusion we discuss the properties and peculiarities of Béguin's [7] analytic interpretation of systems with servo-constraints; we also discuss the difference between the ordinary and the parametric constraints and servo-constraints.

1. We consider a system of material points with masses m_ν $(\nu = 1, \ldots, N)$, subjected to certain constraints and moving relative to an inertial coordinate system under the action of forces applied to it. The Cartesian coordinates of the system's points and the projections of the given forces onto the coordinate axes are denoted by x_i and X_i $(i = 1, \ldots, n = 3N)$, so that the coordinates of the ν-th point, its mass and the coordinate axes projections of the active forces on it are $x_{3\nu-2}$, $x_{3\nu-1}$, $x_{3\nu}$, $m_{3\nu-2} = m_{3\nu-1} = m_{3\nu}$, $X_{3\nu-2}$, $X_{3\nu-1}$, $X_{3\nu}$, respectively. The total time derivatives are denoted by an upper right dot ($x^\cdot \equiv dx / dt$). The specified forces X_i are taken to be known functions of coordinates x_j, of the velocities $x_j^\cdot$ $(j = 1, \ldots, n)$, of time t and, perhaps, of certain variable parameters u_r $(r = 1, \ldots, k)$.

Let the system be subject to the geometric constraints

$$f_s(x_1, \ldots, x_n, u_1, \ldots, u_k, t) = 0 \quad (s = 1, \ldots, m_1) \tag{1.1}$$

and to the kinematic nonintegrable constraints

$$\varphi_p(x_1, \ldots, x_n, x_1^\cdot, \ldots, x_n^\cdot, u_1, \ldots, u_k, t) = 0 \; (p = 1, \ldots, m_2) \tag{1.2}$$

nonlinear, in the general case, relative to the velocities $x_i^\cdot$, where the number of constraints $m = m_1 + m_2 < n$. Generally the constraint equations (1.1) and (1.2) are assumed dependent on the parameters u_r, which does not exclude, it is understood, the presence among them of ordinary constraints not depending upon parameters u_r. Constraints (1.1) and linear constraints (1.2), depending on variable parameters, were apparently first examined by Grdina [1, 2] who called them volitional constraints and volitional parameters, respectively. Later on Kirgetov [3, 4] called them parametric constraints and he called the parameters u_r the control parameters. Here we adopt the later designations.

We shall assume that the variable parameters u_r, not defined beforehand, can be given, during the motion, arbitrary values from a specified control region U and, by the same token, can control the system's motion by a suitably specified or chosen control law for the the system [3]. In what follows we limit the analysis to only those variable parameters $u_r \in U$, which possess the first $u_r^\cdot$ and, possibly, the second $u_r^{\cdot\cdot}$ total time derivatives, where for simplicity we assume that the values of the derivatives also belong to the specified control region U, i.e.

$$u_r \in U, \; u_r^\cdot \in U, \quad u_r^{\cdot\cdot} \in U \qquad (r = 1, \ldots, k)$$

Thus, we shall examine the general case when both the given forces (dynamic control) as well as the constraints (kinematic control) depend upon the control parameters.

The functions $X_i(x, x^\cdot, u, t)$ and $\varphi_p(x, x^\cdot, u, t)$ are assumed to belong to function class C_1, while the functions $f_s(x, u, t)$, to class C_2. The constraints (1.1) and (1.2) are reckoned to be compatible and independent for any values of $u_r \in U$. This implies that the functional determinants of functions f_s in the variables x_i and of the functions φ_p in the variables $x_i^\cdot$ have the ranks m_1 and m_2, respectively, for any $u_r \in U$. In addition, we assume the constraints to be independent of the given forces X_i acting on the system. Parametric constraints are a generalization of the ordinary nonstationary constraints and reduce to the latter when instead of the control parameters we substitute into Eqs. (1.1) and (1.2) the relations

$$u_r = u_r(x_1, \ldots, x_n, x_1^\cdot, \ldots, x_n^\cdot, t) \quad (r = 1, \ldots, k) \tag{1.3}$$

by which the system control law can be given in the general case [3] ; in particular, they can be specified as $u_r = u_r(t)$.

The parametric constraints (1. 1), as also the usual geometric constraints, restrict the spatial positions of the system's points. Every position of the system for which the coordinates of its points satisfy Eqs. (1. 1) is called a virtual position for the given instant t and for the given values of parameters $u_r \in U$. In addition, constraints (1. 1), as well as constraints (1. 2), impose specific restrictions both on the velocities $x_i^{\cdot}$ as well as on the accelerations $x_i^{\cdot\cdot}$ of the system's points. As a matter of fact, the equations of bilateral constraints (1. 1) must be satisfied at any instant; therefore, the total time derivatives on the left-hand sides of Eqs. (1. 1) must equal zero

$$\frac{df_s}{dt} = \sum_i \frac{\partial f_s}{\partial x_i} x_i^{\cdot} + \sum_r \frac{\partial f_s}{\partial u_r} u_r^{\cdot} + \frac{\partial f_s}{\partial t} = 0 \qquad (s = 1, \ldots, m_1) \tag{1.4}$$

Relations (1. 4) constrain the velocity components with respect to the gradients $\operatorname{grad}_x f_s$ of the finite constraints; then, as also (1. 2), depend not only on the values of t, x_i, $x_i^{\cdot}$, but also of u_r and $u_r^{\cdot}$, if there are parametric constraints among the constraints (1. 1).

Every collection of velocities $x_i^{\cdot}$, satisfying conditions (1. 2) and (1. 4) for a given system position x_i, virtual for the instant t being considered and for the values of parameters $u_r \in U$, and for the given values of $u_r^{\cdot} \in U$, is called a system of kinematically virtual velocities. In exactly the same way, by differentiating Eqs. (1. 4) and (1. 2) with respect to t, we obtain the conditions

$$\frac{d^2 f_s}{dt^2} = \sum_i \left(\frac{\partial f_s}{\partial x_i} x_i^{\cdot\cdot} + x_i^{\cdot} \frac{d}{dt}\frac{\partial f_s}{\partial x_i}\right) + \sum_r \left(\frac{\partial f_s}{\partial u_r} u_r^{\cdot\cdot} + u_r^{\cdot} \frac{d}{dt}\frac{\partial f_s}{\partial u_r}\right) + \frac{d}{dt}\frac{\partial f_s}{\partial t} = 0 \qquad (s = 1, \ldots, m_1) \tag{1.5}$$

$$\frac{d\varphi_p}{dt} = \sum_i \left(\frac{\partial \varphi_p}{\partial x_i^{\cdot}} x_i^{\cdot\cdot} + \frac{\partial \varphi_p}{\partial x_i} x_i^{\cdot}\right) + \sum_r \frac{\partial \varphi_p}{\partial u_r} u_r^{\cdot} + \frac{\partial \varphi_p}{\partial t} = 0 \qquad (p = 1, \ldots, m_2)$$

for the accelerations $x_i^{\cdot\cdot}$ of the system's points. These conditions depend not only on the values of t, x_i, $x_i^{\cdot}$, u_r, but also of $u_r^{\cdot}$ and of $u_r^{\cdot\cdot}$ if there are parametric constraints among constraints (1. 1).

Every collection of accelerations $x_i^{\cdot\cdot}$, satisfying conditions (1. 5) for given positions x_i and velocities $x_i^{\cdot}$ of the system points, virtual for the instant t being considered and for the values of u_r, $u_r^{\cdot} \in U$, and for the given values of $u_r^{\cdot\cdot} \in U$ is called a system of kinematically virtual accelerations. Infinitely small displacements

$$\Delta x_i = x_i(t + dt) - x_i(t) = x_i^{\cdot} dt + {}^1/_2 x_i^{\cdot\cdot} (dt)^2 + \ldots \qquad (i = 1, \ldots, n)$$

which the system's points can accomplish in infinitesimal time intervals dt from a given position $x_i(t)$, corresponding to some system of kinematically virtual velocities $x_i^{\cdot}$ and accelerations $x_i^{\cdot\cdot}$ for the instant t being considered and for the values of u_r, $u_r^{\cdot}$, $u_r^{\cdot\cdot} \in U$ are called the kinematically virtual displacements of the system.

Suppose that at a given instant t the system takes some virtual position defined by the coordinates $x_i\,(t)$ of its points. Let us consider any two kinematically virtual displacements Δx_i and $\Delta x'_i$ of the system's points, accomplished in one and the same infinitesimal interval dt from one and the same given position and corresponding to two systems of virtual velocities $x_i^{\cdot}$ and $x_i'^{\cdot}$ and accelerations $x_i^{\cdot\cdot}$ and $x_i'^{\cdot\cdot}$ for given $t,\ u_r,\ u_r^{\cdot},\ u_r^{\cdot\cdot}$, and let us consider their difference

$$\Delta x_i' - \Delta x_i = \Delta x_i^{\cdot} dt + {}^1/_2 \Delta x_i^{\cdot\cdot}\,(dt)^2 + \ldots \quad (i = 1, \ldots, n) \tag{1.6}$$
$$(\Delta x_i^{\cdot} = x_i'^{\cdot} - x_i^{\cdot}, \quad \Delta x_i^{\cdot\cdot} = x'^{\cdot\cdot}_i - x_i^{\cdot\cdot})$$

The collection of principal parts (of one and the same order of smallness relative to dt for all i) of these differences is called, as in the case of ordinary constraints [5], the virtual displacement of the system and is denoted $\delta x_i\ (i = 1, \ldots, n)$. When not all $\Delta x_i^{\cdot} = 0\ (i = 1, \ldots, n)$, the virtual displacements of the system's points are determined by the formulas

$$\delta x_i = \Delta x_i^{\cdot} dt \quad (i = 1, \ldots, n) \tag{1.7}$$

If, however, all $\Delta x_i^{\cdot} = 0\ (i = 1, \ldots, n)$, then the virtual displacements are

$$\delta x_i = {}^1/_2 \Delta x_i^{\cdot\cdot}\,(dt)^2 \quad (i = 1, \ldots, n) \tag{1.8}$$

Thus, the system's virtual displacements are elementary displacements of the points, admissible under the constraints at a given instant t and satisfying the conditions

$$\sum_i \frac{\partial f_s}{\partial x_i}\,\delta x_i = 0, \quad \sum \frac{\partial \varphi_p}{\partial x_i^{\cdot}}\,\delta x_i = 0 \tag{1.9}$$
$$(s = 1, \ldots, m_1;\ p = 1, \ldots, m_2)$$

Under the above-mentioned assumption on the independence of constraints (1.1) and (1.2) Eqs. (1.9), obviously, are independent.

Constraints (1.1) and (1.2) are taken to be ideal, i.e. such that their reactions R_i to every virtual displacement (1.9) of the system equal zero

$$\sum_i R_i \delta x_i = 0 \tag{1.10}$$

A consequence of the axiom (1.10) defining ideal constraints is, as in the case of the usual constraints, the fundamental principle of the dynamics of controlled systems, namely, the d'Alembert-Lagrange principle [1, 3]

$$\sum_i (m_i w_i - X_i)\,\delta x_i = 0 \tag{1.11}$$

being a variational differential principle valid for the true motion of the system with accelerations $x_i^{\cdot\cdot} = w_i$ for any infinitesimal virtual displacements δx_i from a given configuration of the system in its true motion. The equations of motion of controlled systems, in the form of equations with multipliers, of the equations of Lagrange, Hamilton, Jacobi and Appel, were derived in [1 — 4] from relations (1.11).

2. Let us show that the fundamental variational differential principles of dynamics, equivalent to the d'Alembert-Lagrange principle, are valid for controlled systems.

1°. The Jourdin principle. For given instants t and values of $u_r,\ u_r^{\cdot} \in U$ we take as given the system's configuration x_i and its true motion with velocities $x_i^{\cdot} = v_i$ and we consider some kinematically virtual motion with the same values of x_i and with infinitely close velocities $x_i^{\boldsymbol\cdot} = v_i + \delta x^{\boldsymbol\cdot}_i$. From Eqs. (1. 9), with due regard to (1. 7), it follows that the quantities $\delta x_i^{\boldsymbol\cdot}$ satisfy the conditions [5]

$$\sum_i \frac{\partial f_s}{\partial x_i}\,\delta x_i^{\boldsymbol\cdot} = 0, \quad \sum_i \frac{\partial \varphi_p}{\partial x_i^{\boldsymbol\cdot}}\,\delta x_i^{\boldsymbol\cdot} = 0 \qquad (s = 1, \ldots, m_1,\ p = 1, \ldots, m_2) \tag{2. 1}$$

Since conditions (2. 1) for $\delta x_i^{\boldsymbol\cdot}$ coincide with conditions (1. 9) for δx_i, we can write Eq. (1. 1) as

$$\sum_i (m_i w_i - X_i)\,\delta x_i^{\boldsymbol\cdot} = 0 \tag{2. 2}$$

Equation (2. 2) expresses the variational differential Jourdin principle, namely, Eq. (2. 2) is valid for any $\delta x_i^{\boldsymbol\cdot}$ satisfying conditions (2. 1) for a true motion in the class of motions conceivable in the sense of Jourdin (kinematically virtual motions satisfying the conditions imposed on the system by the constraints and the conditions of the constancy of the x_i for given instants t and values of u_r, $u_r^{\boldsymbol\cdot} \in U$) .

Example 2. 1. From relations (2. 2) and (2. 1) we can derive the equations of motion of the system in various forms; for instance, as equations with multipliers [1]

$$m_i w_i = X_i + \sum_s \lambda_s \frac{\partial f_s}{\partial x_i} + \sum_p \mu_p \frac{\partial \varphi_p}{\partial x_i^{\boldsymbol\cdot}} \quad (i = 1, \ldots, n) \tag{2. 3}$$

to which we have to add on the constraint Eqs. (1. 1) and (1. 2). Here λ_s and μ_p are undetermined Lagrange multipliers. The system of Eqs. (2. 3), (1. 1) and (1. 2) for x_i, λ_s and μ_p is not closed since besides the $n + m_1 + m_2$ unknowns it also contains the control parameters u_r. To close the system we need to specify [3] or, from some additional conditions, determine the control law (1. 3). As we see from Eqs. (2. 3) the constraints (1. 1) and (1. 2) force their own reactions on the system, depending, in the general case, on the control parameters, so that in the final analysis the kinematic control reduces to a dynamic control, to control by forces. The undetermined multipliers can be eliminated from (2. 3) if by these equations we replace $x_i^{\boldsymbol\cdot\boldsymbol\cdot}$ in Eqs. (1. 5) and solve the system of inhomogeneous equations with a nonzero determinant thus obtained relative to λ_s and μ_p. As a result we find the latter in the form of certain functions of the variables t, x_i, $x_i^{\boldsymbol\cdot}$, u_r, $u_r^{\boldsymbol\cdot}$, $u_r^{\boldsymbol\cdot\boldsymbol\cdot}$ and, substituting into (2. 3), obtain the system of equations

$$m_i x_i^{\boldsymbol\cdot\boldsymbol\cdot} = \Phi_i\,(x, x^{\boldsymbol\cdot}, u, u^{\boldsymbol\cdot}, u^{\boldsymbol\cdot\boldsymbol\cdot}, t) \quad (i = 1, \ldots, n) \tag{2. 4}$$

whose right-hand sides depend, in the general case, not only on u_r but also on $u_r^{\boldsymbol\cdot}$ and $u_r^{\boldsymbol\cdot\boldsymbol\cdot}$. In special cases, when there are no parametric ones among the geometric constraints (1. 1), the right-hand sides of Eqs. (2. 4) are independent of $u_r^{\boldsymbol\cdot\boldsymbol\cdot}$, while if there are no parametric ones among the nonintegrable constraints (1. 2), they will be independent of $u_r^{\boldsymbol\cdot}$ These same properties are possessed by the equations of motion in generalized coordinates [1, 2] and in quasi-coordinates [4] which are more convenient than equations in Cartesian coordinates in applications with a large number of variables.

Owing to the dependence of the equations of motion not only on u_r, as for dynamic control, but also on $u_r^{\boldsymbol\cdot}$ and $u_r^{\boldsymbol\cdot\boldsymbol\cdot}$, the kinematic control is somewhat more complicated than dynamic control, but at the same time permits greater possibilities for control. Obviously, general control theory [8] is applicable to kinematically controlled systems too. If we multiply Eqs. (2. 3) by the real variables $dx_i = v_i dt$ and sum over all i, we obtain the equation

$$dT = \sum_i X_i\,dx_i + \sum_{i,\,s} \lambda_s \frac{\partial f_s}{\partial x_i^{\cdot}}\,dx_i + \sum_{i,\,p} \mu_p \frac{\partial \varphi_p}{\partial x_i^{\cdot}}\,dx_i \tag{2.5}$$

expressing a theorem on the system's kinetic energy. When the active forces are independent of time, of the parameters u_r and of the potential forces, Eq. (2.5) becomes

$$d(T-U) = \sum_i \Big(\sum_s \lambda_s \frac{\partial f_s}{\partial x_i^{\cdot}} + \sum_p \mu_p \frac{\partial \varphi_p}{\partial x_i^{\cdot}}\Big)\,dx_i \tag{2.6}$$

$$\Big(\sum_i X_i\,dx_i = dU(x_1, \ldots, x_n)\Big)$$

The reactions of the parametric forces on the real variables is not zero, in general; therefore, the total mechanical energy $T-U$ of the system does not remain constant. As in the case of servo-constraints [7] the reactions, depending upon their sign, lead to an increase or a decrease of the system energy and, in particular, can damp the oscillations of a system in which energy dissipation is lacking.

2°. The Mach inequalities and the generalized Gauss principle. At an instant t and for given values of parameters u_r, and of their velocities $u_r^{\cdot}$ and accelerations $u_r^{\cdot\cdot}$ from domain U we assume as given the system's state x_i and $x_i^{\cdot} = v_i$ in some real motion of it with accelerations $w_i = dx_i^{\cdot}/dt$ and we consider some kinematically virtual motion with the same x_i and $x_i^{\cdot}$ and with infinitely close accelerations $\delta x_i^{\cdot} / dt = w_i + \delta x_i^{\cdot}$ With due regard to (1.8), from Eqs. (1.9) we obtain the conditions [5]

$$\sum_i \frac{\partial f_s}{\partial x_i^{\cdot}}\,\delta x_i^{\cdot\cdot} = 0, \qquad \sum_i \frac{\partial \varphi_p}{\partial x_i^{\cdot}}\,\delta x_i^{\cdot\cdot} = 0, \qquad (s = 1, \ldots, m_1,\ p = 1, \ldots, m_2) \tag{2.7}$$

for $\delta x_i^{\cdot\cdot}$. Comparing (2.7) with Eqs. (1.9) we see that a difference of accelerations is found among the virtual displacements of the system. Consequently, in this case Eq. (1.11) can be written as

$$\sum_i (m_i w_i - X_i)\,\delta x_i^{\cdot\cdot} = 0$$

At instant t we free the system of a part of the constraints imposed on it and by $\partial x_i^{\cdot} / dt$, we denote the accelerations of the system's points in a real free motion under the action of those same forces X_i. Since among the virtual displacements of the free system we can find the virtual displacements of a system with constraints, an equation of form (1.11) for the former can be written as

$$\sum_i \Big(m_i \frac{\partial x_i^{\cdot}}{dt} - X_i\Big)\,\delta x_i^{\cdot\cdot} = 0$$

Subtracting this relation from the one preceding, we obtain the equality [6]

$$A_{d\delta} + A_{d\partial} - A_{\delta\partial} = 0 \tag{2.8}$$

where the measure of deviation of the real (d) motion from the conceivable (δ) one is

$$A_{d\delta} = \frac{1}{2}\sum m_i\left(\frac{dx_i^{\boldsymbol{\cdot}}}{dt} - \frac{\delta x_i^{\boldsymbol{\cdot}}}{dt}\right)^2$$

The quantities $A_{d\partial}$ and $A_{\delta\partial}$ are determined analogously. The Mach inequalities for a controlled system follow from equality (2.8),

$$A_{d\delta} < A_{\partial\delta}, \quad A_{d\partial} < A_{\partial\delta} \tag{2.9}$$

The second one of these inequalities is an expression of the generalized Gauss principle, namely, the measure of the deviation of the real (d) motion of a system from the real (∂) motion of the system freed of a part of the constraints is less than the measure of the deviation of the latter from the conceivable (δ) motion of the system. If the system is freed of all constraints, the second inequality in (2.9) reduces to the Gauss principle for controlled systems; for the real motion of a system the constraint

$$Z = \frac{1}{2}\sum_i m_i\left(x_i \quad - \frac{X_i}{m_i}\right)^2 \tag{2.10}$$

has a minimum in the class of Gauss-conceivable accelerations satisfying Eqs. (1.5) with given values of $t,\ x_i,\ x_i^{\boldsymbol{\cdot}},\ u_r,\ u_r^{\boldsymbol{\cdot}},\ u_r^{\boldsymbol{\cdot\cdot}}$.

Note 2.1. The derivation we have presented of the Gauss principle from the d'Alembert-Lagrange principle attests to the equivalence of these principles for controlled systms. It also corroborates the indirect proof [1] of the validity of the Gauss principle for systems with parametric constraints and refutes the assertion [3] on the absence of this principle for such systems. Such an assertion was made in [3] on the basis of an analysis of the equations for kinematically virtual motions, obtained by substituting control law (1.3) into Eqs. (1.1), whereas the virtual displacements were determined by the first group of Eqs. (1.9). Such an approach, however, is inconsistent since if we can really determine the kinematically virtual motions with due regard to the control law, then with due regard to the latter we should be able to determine the virtual displacements too; in this connection, obviously, the parametric constraints become ordinary constraints. We see as well that the modification of the Gauss principle (replacing in (2.10) the total accelerations of the points by their components tangential to the constraints), proposed in [3], is equivalent to the Gauss principle not only for ordinary holonomic constraints [3] but also for parametric constraints (1.1) and (1.2).

3°. Chetaev's principle. If we repeat the arguments in [9], we can easily see that the modification of the Gauss principle, given by Chetaev, is also valid for systems with parametric constraints, namely, the operation on an elementary cycle consisting of the direct Gauss-conceivable motion in the effective force field and of the motion retrograde (inverse) in the force field which would be sufficient for the creation of the real motion if the controlled mechanical system were perfectly free, has a maximum for the real motion.

As follows from what we presented in Sect. 2 the differential principles of Jourdin, Gauss and Chetaev are equivalent to the d'Alembert-Lagrange principle and are applicable for controlled systems with parametric constraints (1.1) and (1.2). The dynamics of controlled systems can be founded on each of these principles.

3. In conclusion we discuss briefly the properties of a class of controlled systems, namely, systems with servo-constraints, whose general theory was developed by Béguin [7] (also see the almost verbatim reproduction of this theory in Ref. [10]), and also systems with so -called conditional constraints [11], and we compare the reactions of servo-constraints with the reactions of ordinary and parametric constraints. The usual constraints analyzed in mechanics express the conditions for the contacts between bodies both occurring as well as not occurring in the system, where the latter either are immovable or are in a motion preassigned in time. The reaction forces of such constraints are, obviously, contact action forces which, according to a well-known classification, relate to the category of passive forces in contrast to the category of given or active forces applied to the system, on which the reaction forces depend.

Parametric constraints also express contact conditions, but in contrast to the ordinary constraints their reactions are not purely passive forces since they depend not only on the active forces X_i, but, in general, also on the control parameters occurring in Eqs. (1.1) and (1.2), which have a possible active effect on the system's motion.

We begin with axiom (1.10) for the reactions of ideal constraints, ordinary and parametric ones. Béguin [7] noted that mechanisms exist which realize constraints by a method entirely different from the one indicated, for which it is impossible to digress from the method of realizing constraints. The realization of constraints by them is ensured not with the aid of a simple contact, i.e., not passively, but with the use of auxiliary energy sources which automatically go into action and are automatically controlled so that the given constraint is realized continuously. Béguin called such constraints servo-constraints or constraints of the second kind, in distinction to ordinary constraints or constraints of the first kind, this designation also applied to parametric constraints.

The reaction forces Φ_i of the servo-constraints, applied to the system's points, can be forces acting at a distance (for example, electro-magnetic or other forces), internal stresses yielding a contraction or an expansion of the bodies (compressed air, muscles of a living being, etc.), contact forces of foreign bodies whose position depends upon a number of coordinates of the system and whose motion is automatically controlled so as to realize the given constraint; the contact forces depend both on the contact constraints as well as on the servo-constraints. Just as the reactions R_i of constraints of the first kind, the reactions Φ_i of the servo-constraints are not known in advance; only the values they must take in order to realize the given servo-constraint are known. However, the forces Φ_i are caused, as we have already noted, by the presence of additional energy sources and in this sense belong to the category of active forces; the servo-constraints depend upon the action of such forces.

In contrast to axiom (1.10) the reactions of ideal constraints of the first kind are the sum of the elementary reactions of servo-constraints on the virtual displacements and, in general, are nonzero; a consequence of this is that the d'Alembert-Lagrange principle for systems with servo-constraints can be written as

$$\sum_i (m_i w_i - X_i - \Phi_i)\,\delta x_i = 0 \qquad (3.1)$$

in contrast to the form (1.11) of this principle for systems with ideal constraints of the first kind. This is what causes the analytic distinction of systems with servocontraints from systems with ordinary or parametric constraints. By assuming that among the virtual displacements of the system, admissible under constraints of the first kind. there are displacements of the form

$$\sum_i a_{\varkappa i}\delta x_i = 0 \quad (\varkappa = 1, \ldots, j) \tag{3.2}$$

for which the reactions of the servo-constraints equal zero by virtue of the very way in which they act, Béguin writes the d'Alembert-Lagrange principle for such displacements in the usual form (1.11) and derives the equations of motion of systems with servo constraints, to which it is necessary to add on the equations of the servo-constraints. The problem turns out to be well-defined if the number of restrictive conditions (3.2) equals the number of servo-constraints. The equations of motion of a system with servo-constraints can take the form of the Lagrange or the Appel equations [7]. It is remarkable that when the reactions of the servo-constraints consist exclusively of the reactions of moving bodies whose positions depend upon a certain number of the system's coordinates, equal to the number of servo-constraints, the problem's solution is independent of the inertia of these bodies and of the forces applied to them. In exactly the same way, if in the system we can separate two parts Σ and Σ_1, such that no servo-constraint reactions besides the reactions of system Σ_1, act on system Σ and the number of coordinates on which the latter depend equals the number of servo-constraints, then the inertia of system Σ_1 and the forces applied to it do not affect the motion of system Σ. In similar cases we restrict ourselves to compiling the equations of motion of only the system Σ if we are not interested in the servo-constraint reactions.

We note that from the d'Alembert-Lagrange principle for virtual displacements satisfying conditions (3.2) for systems with servo-constraints we can derive [12] the Mach inequalities and the generalized Gauss principle without the servo-constraint reactions occurring explicitly in it.

The following problem [11] is close to the problem of the motion of systems with servo-constraints. Suppose that we are given a number of relations

$$\begin{aligned} &\Phi_\sigma (x_1, \ldots, x_n, u_1, \ldots, u_k, t) = 0 \ (\sigma = 1, \ldots, h) \\ &\Phi_{h+\pi} (x_1, \ldots, x_n, x_i^{\boldsymbol{\cdot}}, \ldots, x_n^{\boldsymbol{\cdot}}, u_1, \ldots, u_k, t) = 0 \ (\pi = 1, \ldots, g) \end{aligned} \tag{3.3}$$

and that an exact satisfaction of these relations during the motion is required by using an appropriate control of the system. Relations (3.3) are called [11] conditional constraints; their reactions must be identically equal to zero. Two stages are recommended in [11] for solving this problem: 1) set up the equations of motion of the system with due regard to all its constraints, both the actual (1.1) and (1.2) as well as the conditional (3.3), treating the latter as actual; 2) free the system of the conditional constraints by complementing the collection of its generalized coordinates and quasi-coordinates by the necessary number of new coordinates and quasi-coordinates and set up the equations of motion of the freed system, corresponding to the new coordinates and quasi-coordinates, in which the equations of the conditional constraints are then formally accounted for. The equations in the first group thus obtained are called the equations of motion of the system with conditional constraints, while in the second group the dynamic conditions for the fulfillment of the conditional constraints, "taking into account that, in the final analysis, these are conditions on the forces in order that the conditional constraints be satisfied" (see [11]).

Note 3.1. Such a terminology is not fully successful since by example we have shown below that the result can prove to be contradictory; the equations in the second group yield the equations of motion, while in the first group conditions on the controls. Actually, it is advisable to restrict ourselves to one stage namely, set up the equations of motion of the system with due regard only to constraints (1.1) and (1.2) and then to account for Eqs. (3.3) in the resulting equations, thus obtaining both the equations of motion as well as the conditions on the controls.

In [11] the conditional constraints were identified with Béguin's servo-constraints since one or the other "are realized as if compulsorily by means of a suitable control of the system". Although the latter is true, nevertheless, there is a subtle difference between servo-constraints and conditional constraints in their analytic treatment. In [7] servo-constraints are interpreted as precisely the constraints whose reactions Φ_i are unknown in advance, but by virtue of the very method of realizing them the displacements (3.2) on which the servo-constraints do not react are known. Thanks to this, Béguin succeeded in obtaining the equations of motion of a system with servo-constraints without the servo-constraint reactions Φ_i explicitly occurring in them, as well as in the case of ideal constraints of the first kind. In Kirgetov's interpretation, proceeding from d'Alembert-Lagrange principle in its usual form (1.11), it is assumed that "the satisfaction of the conditional constraints is achieved exclusively at the expense of external active forces acting on the system and of the reactions of real parametric constraints" (see [11]). In other words, it is assumed that the expressions for forces Φ_i are known, referred not to the number of reactions but to the number of external active forces X_i with whose aid relations (3.3) are realized, thanks to which relation (3.1) can be written in the form (1.11). When the expression for forces Φ_i applicable to the system for realizing the servo-constraints are not know beforehand, these forces can be set among the given forces X_i only purely formally, with a subsequent compulsory determination of their expressions from the equations obtained from the d'Alembert-Lagrange principle in the form (1.11).

Note 3.2. In essence both the servo-constraints as well as the conditional constraints are invariant relations of the equations of motion of the controlled systems, first determined by Poincaré [13] for autonomous systems of differential equations not containing control parameters. Obviously, for controlled systems there is a greater possibility for the existence of invariant relations thanks to the possibility of making a suitable choice of the control parameters. The distinction between servo-constraints and conditional constraints in this connection is the fact that the first are realized by forces Φ_i additional to the given active forces X_i, while the second, by only the given active forces X_i and the parametric constraints. However, we should bear in mind the possibility of the existance of invariant relations for an uncontrolled system.

Example 3.1. Let us illustrate what we have presented above by analyzing the solution of Béguin's problem on the planar motion of a plate hinged to a circular disk (we retain all the notation of [7]). Relying on the theory he developed Béguin applies the Lagrange equation relative to the plate separately, and obtains the following equation of its motion:

$$M\,(b^2 + k^2)\,\beta^{\cdot\cdot} - MRb\beta^{\cdot 2} + Fa\sin\beta = 0 \tag{3.4}$$

noting that if the constraint $\alpha - \beta = \pi/2$ were realized directly by the contact of the plate and the disk, then the system's motion would be determined by the equation

$$[M\,(R^2 + b^2 + k^2) + I_1]\,\beta^{\cdot\cdot} + F\,(a\sin\beta + R\cos\beta) = 0 \tag{3.5}$$

By examining Béguin's problem from the point of view developed in [11], of interpreting servo-constraints as conditional constraints, Kirgetov concluded that Béguin "was not correct in the given case" and stated that the equation of motion is

$$[M(R^2 + b^2 + k^2) + I_1]\beta^{\cdot\cdot} + F(a \sin\beta + R\cos\beta) = u \qquad (3.6)$$

which differs from Eq. (3.5) only in the right-hand side, while Eq. (3.4) "is a dynamic condition for the fulfillment of the conditional constraint imposed on the system". Actually, however, Béguin is correct. As a matter of fact, Eq. (3.4) cannot serve as the dynamic condition for the fulfillment of the conditional constraint since it does not contain the control u. Equation (3.6) would be the system's equation of motion if its right-hand side were a given function. However, it is evident that the conditional constraint being examined cannot be realized for every given moment u. In fact, Eq. (3.6) can serve for determining an expression for moment u, which is easily obtained by substituting the conditional constraint into Eq. (3.6) and allowing for the Eq. (3.4) of motion of the plate.

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